

ON INVARIANT SUBMANIFOLDS OF A GENERALIZED KENMOTSU MANIFOLD

AYSEL TURGUT VANLI AND RAMAZAN SARI

ABSTRACT. In this paper, invariant submanifolds of a generalized Kenmotsu manifold are studied. Necessary and sufficient conditions are given on a submanifold of a generalized Kenmotsu manifold to be an invariant submanifold. In this case, we investigate further properties of invariant submanifolds of a generalized Kenmotsu manifold. In addition, some theorems are given related to an invariant submanifold of a generalized Kenmotsu manifold. We consider semiparallel and 2-semiparallel invariant submanifolds of a generalized Kenmotsu manifold.

1. INTRODUCTION

In 1963, Yano [10] introduced an f -structure on a C^∞ m -dimensional manifold M , defined by a non-vanishing tensor field φ of type $(1, 1)$ which satisfies $\varphi^3 + \varphi = 0$ and has constant rank r . It is known that in this case r is even, $r = 2n$. Moreover, TM splits into two complementary subbundles $Im\varphi$ and $\ker\varphi$ and the restriction of φ to $Im\varphi$ determines a complex structure on such subbundle. It is known that the existence of an f -structure on M is equivalent to a reduction of the structure group to $U(n) \times O(s)$ [2], where $s = m - 2n$.

In [7], K. Kenmotsu has introduced a Kenmotsu manifold. In [8], present authors have introduced generalized Kenmotsu manifolds.

In [4], J. Deprez defined a semi-parallel immersion. Let $i : M \rightarrow \bar{M}$ be an isometric immersion of a Riemannian manifold, \bar{R} curvature tensor of the Van der Waerden-Bortolotti connection $\bar{\nabla}$ of i and h the second fundamental form of i , then i is said to be semi-parallel if $\bar{R}.h = 0$. In addition, J. Deprez studied semi-parallel hypersurfaces in [5].

In [1], K. Arslan and colleagues defined, a submanifold to be 2-semi-parallel if $R.\nabla h = 0$ where R curvature tensor of the Van der Waerden-Bortolotti connection ∇ of i and h the second fundamental form of i .

In this paper, we give necessary and sufficient conditions for a submanifold of a generalized Kenmotsu manifold to be an invariant submanifold and we consider the invariant case. In addition, we study semi-parallel and 2-semi-parallel invariant submanifolds of a generalized Kenmotsu manifold.

2. GENERALIZED KENMOTSU MANIFOLDS

In [6], a $(2n+s)$ -dimensional differentiable manifold M is called metric f -manifold if there exist an $(1, 1)$ type tensor field φ , s vector fields ξ_1, \dots, ξ_s , s 1-forms

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η^1, \dots, η^s and a Riemannian metric g on M such that

$$(2.1) \quad \varphi^2 = -I + \sum_{i=1}^s \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta_{ij}$$

$$(2.2) \quad g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^s \eta^i(X) \eta^i(Y),$$

for any $X, Y \in \Gamma(TM)$, $i, j \in \{1, \dots, s\}$. In addition, we have

$$(2.3) \quad \eta^i(X) = g(X, \xi_i), \quad g(X, \varphi Y) = -g(\varphi X, Y), \quad \varphi \xi_i = 0, \quad \eta^i \circ \varphi = 0.$$

Then, a 2-form Φ is defined by $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$, called the *fundamental 2-form*. Moreover, a framed metric manifold is *normal* if

$$[\varphi, \varphi] + 2 \sum_{i=1}^s d\eta^i \otimes \xi_i = 0,$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ .

In [9], let M be a $(2n+s)$ -dimensional metric f-manifold. If there exists 2-form Φ such that $\eta^1 \wedge \dots \wedge \eta^s \wedge \Phi^n \neq 0$ on M then M is called an almost s -contact metric structure.

Definition 1. Let M be an almost s -contact metric manifold of dimension $(2n+s)$, $s \geq 1$, with $(\varphi, \xi_i, \eta^i, g)$. M is said to be a generalized almost Kenmotsu manifold if for all $1 \leq i \leq s$, 1-forms η^i are closed and $d\Phi = 2 \sum_{i=1}^s \eta^i \wedge \Phi$. A normal generalized almost Kenmotsu manifold M is called a generalized Kenmotsu manifold [8].

Example 1. Let $n = 2$ and $s = 3$. The vector fields

$$\begin{aligned} e_1 &= f_1(z_1, z_2, z_3) \frac{\partial}{\partial x_1} + f_2(z_1, z_2, z_3) \frac{\partial}{\partial y_1}, \\ e_2 &= -f_2(z_1, z_2, z_3) \frac{\partial}{\partial x_1} + f_1(z_1, z_2, z_3) \frac{\partial}{\partial y_1}, \\ e_3 &= f_1(z_1, z_2, z_3) \frac{\partial}{\partial x_2} + f_2(z_1, z_2, z_3) \frac{\partial}{\partial y_2}, \\ e_4 &= -f_2(z_1, z_2, z_3) \frac{\partial}{\partial x_2} + f_1(z_1, z_2, z_3) \frac{\partial}{\partial y_2}, \\ e_5 &= \frac{\partial}{\partial z_1}, e_6 = \frac{\partial}{\partial z_2}, e_7 = \frac{\partial}{\partial z_3} \end{aligned}$$

where f_1 and f_2 are given by

$$\begin{aligned} f_1(z_1, z_2, z_3) &= c_2 e^{-(z_1+z_2+z_3)} \cos(z_1 + z_2 + z_3) - c_1 e^{-(z_1+z_2+z_3)} \sin(z_1 + z_2 + z_3), \\ f_2(z_1, z_2, z_3) &= c_1 e^{-(z_1+z_2+z_3)} \cos(z_1 + z_2 + z_3) + c_2 e^{-(z_1+z_2+z_3)} \sin(z_1 + z_2 + z_3) \end{aligned}$$

for nonzero constant c_1, c_2 . It is obvious that $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ are linearly independent at each point of M . Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

for all $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$ and given by the tensor product

$$g = \frac{1}{f_1^2 + f_2^2} \sum_{i=1}^2 (dx_i \otimes dx_i + dy_i \otimes dy_i) + dz_1 \otimes dz_1 + dz_2 \otimes dz_2 + dz_3 \otimes dz_3,$$

where $\{x_1, y_1, x_2, y_2, z_1, z_2, z_3\}$ are standard coordinates in \mathbb{R}^7 . Let η^1, η^2 and η^3 be the 1-form defined by $\eta^1(X) = g(X, e_5)$, $\eta^2(X) = g(X, e_6)$ and $\eta^3(X) = g(X, e_7)$, respectively, for any vector field X on M and φ be the $(1, 1)$ tensor field defined by

$$\begin{aligned} \varphi(e_1) &= e_2, & \varphi(e_2) &= -e_1, & \varphi(e_3) &= e_4, & \varphi(e_4) &= -e_3, \\ \varphi(e_5) &= \xi_1 = 0, & \varphi(e_6) &= \xi_2 = 0, & \varphi(e_7) &= \xi_3 = 0. \end{aligned}$$

Therefore, the essential non-zero component of Φ is

$$\Phi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) = -\frac{1}{f_1^2 + f_2^2} = -\frac{2e^{2(z_1+z_2+z_3)}}{c_1^2 + c_2^2}, \quad i = 1, 2$$

and hence

$$\Phi = -\frac{2e^{2(z_1+z_2+z_3)}}{c_1^2 + c_2^2} \sum_{i=1}^2 dx_i \wedge dy_i.$$

Consequently, the exterior derivative $d\Phi$ is given by

$$d\Phi = -\frac{4e^{2(z_1+z_2+z_3)}}{c_1^2 + c_2^2} (dz_1 + dz_2 + dz_3) \wedge \sum_{i=1}^2 dx_i \wedge dy_i.$$

Since $\eta^1 = dz_1$, $\eta^2 = dz_2$ and $\eta^3 = dz_3$, we find

$$d\Phi = 2(\eta^1 + \eta^2 + \eta^3) \wedge \Phi.$$

In addition, Nijenhuis tensor of φ is different from zero.

Theorem 1. Let $(M, \varphi, \xi_i, \eta^i, g)$ be an almost s -contact metric manifold. M is a generalized Kenmotsu manifold if and only if

$$(2.4) \quad (\bar{\nabla}_X \varphi)Y = \sum_{i=1}^s \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\}$$

for all $X, Y \in \Gamma(TM)$, $i \in \{1, 2, \dots, s\}$, where $\bar{\nabla}$ is Riemannian connection on M [8].

Theorem 2. Let M be a $(2n+s)$ -dimensional generalized Kenmotsu manifold with structure $(\varphi, \xi_i, \eta^i, g)$. Then we have

$$(2.5) \quad \bar{\nabla}_X \xi_j = -\varphi^2 X$$

for all $X, Y \in \Gamma(TM)$, $i \in \{1, 2, \dots, s\}$ [8].

Theorem 3. Let M be a $(2n+s)$ -dimensional generalized Kenmotsu manifold with structure $(\varphi, \xi_i, \eta^i, g)$. Then we have

$$(2.6) \quad \bar{R}(X, Y)\xi_i = \sum_{j=1}^s \{\eta^j(Y)\varphi^2 X - \eta^j(X)\varphi^2 Y\}$$

$$(2.7) \quad \bar{R}(X, \xi_j)\xi_i = \varphi^2 X$$

$$(2.8) \quad \bar{R}(\xi_j, X)\xi_i = -\varphi^2 X$$

$$(2.9) \quad \bar{R}(\xi_k, \xi_j)\xi_i = 0$$

$$(2.10) \quad \bar{R}(\xi_j, X)Y = \sum_{j=1}^s \{g(X, \varphi^2 Y)\xi_j - \eta^j(Y)\varphi^2 X\}$$

for all $X, Y \in \Gamma(TM)$, $i, j, k \in \{1, 2, \dots, s\}$ [8].

3. INVARIANT SUBMANIFOLDS OF A GENERALIZED KENMOTSU MANIFOLD

Let M be a submanifold of the a $(2n + s)$ dimensional generalized Kenmotsu manifold \bar{M} . If $\varphi(T_x M) \subset T_x M$, for any point $x \in M$ and ξ_i are tangent to M for all $i \in \{1, 2, \dots, s\}$, then M is called an invariant submanifold of \bar{M} .

Let ∇ be the Levi-Civita connection of M with respect to the induced metric g . Then Gauss and Weingarten formulas are given by

$$(3.1) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

$$(3.2) \quad \bar{\nabla}_X N = \nabla_X^\perp N - A_N X$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^\perp$. ∇^\perp is the connection in the normal bundle, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N . The second fundamental form h and the shape operator A related by

$$(3.3) \quad g(h(X, Y), N) = g(A_N X, Y).$$

The curvature transformations of M and \bar{M} will be denote by

$$(3.4) \quad R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} \quad X, Y \in \Gamma(TM)$$

and

$$(3.5) \quad \bar{R}(X, Y) = \bar{\nabla}_X \bar{\nabla}_Y - \bar{\nabla}_Y \bar{\nabla}_X - \bar{\nabla}_{[X, Y]} \quad X, Y \in \Gamma(T\bar{M})$$

respectively.

Using (3, 4), (3, 5), the Gauss and the Weingarten formulas, we obtain for any vector fields X, Y and Z tangent to M

$$(3.6) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - A_{h(Y, Z)}(X) + A_{h(X, Z)}(Y) \\ &\quad + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z). \end{aligned}$$

Thus, if W is tangent to M , then we get the Gauss equation

$$(3.7) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) + \bar{g}(h(Y, W), h(X, Z)) \\ &\quad - \bar{g}(h(X, W), h(Y, Z)). \end{aligned}$$

Theorem 4. *Let M be an invariant submanifold of the a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$(3.8) \quad (\nabla_X \varphi)Y = \sum_{i=1}^s \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\},$$

$$(3.9) \quad h(X, \varphi Y) = \varphi h(X, Y)$$

for all $X, Y \in \Gamma(TM)$.

Proof. Since \bar{M} is a generalized Kenmotsu manifold, we have

$$(\bar{\nabla}_X \varphi) Y = \sum_{i=1}^s \{g(\varphi X, Y) \xi_i - \eta^i(Y) \varphi X\}.$$

Using (3, 1) then we have,

$$\begin{aligned} (\bar{\nabla}_X \varphi) Y &= \nabla_X \varphi Y - h(X, \varphi Y) - \varphi(\nabla_X Y - h(X, Y)) \\ &= (\nabla_X \varphi) Y - h(X, \varphi Y) + \varphi h(X, Y). \end{aligned}$$

Comparing the tangential and normal part of last equation, we get the desired result. \square

Corollary 1. *Let M be an invariant submanifold of the a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$(3.10) \quad h(X, \varphi Y) = \varphi h(X, Y) = h(\varphi X, Y),$$

$$(3.11) \quad h(\varphi X, \varphi Y) = -h(X, Y)$$

for all $X, Y \in \Gamma(TM)$.

Theorem 5. *Let M be an invariant submanifold of the a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$(3.12) \quad \nabla_X \xi_j = -\varphi^2 X,$$

$$(3.13) \quad h(X, \xi_j) = 0$$

for all $X, Y \in \Gamma(TM)$.

Proof. For a generalized Kenmotsu manifold using (2, 5), then we get

$$\bar{\nabla}_X \xi_j = X - \sum_{i=1}^s \eta^i(X) \xi_i$$

$$\nabla_X \xi_j + h(X, \xi_j) = X - \sum_{i=1}^s \eta^i(X) \xi_i$$

from the Gauss formula then the equation is implied that

$$\nabla_X \xi_j = X - \sum_{i=1}^s \eta^i(X) \xi_i$$

and

$$h(X, \xi_j) = 0.$$

\square

Theorem 6. *Let M be an invariant submanifold of the a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$(3.14) \quad R(X, Y) \xi_i = \sum_{j=1}^s \{\eta^j(Y) \varphi^2 X - \eta^j(X) \varphi^2 Y\}$$

for all $X, Y \in \Gamma(TM)$.

Proof. Using Gauss equation (3, 6), we have

$$\begin{aligned}\bar{R}(X, Y)\xi_i &= R(X, Y)\xi_i - A_{h(Y, \xi_i)}(X) + A_{h(X, \xi_i)}(Y) + (\nabla_X h)(Y, \xi_i) - (\nabla_Y h)(X, \xi_i). \\ &= R(X, Y)\xi_i - A_{h(Y, \xi_i)}(X) + A_{h(X, \xi_i)}(Y) \\ &\quad + \nabla_X h(Y, \xi_i) - h(\nabla_X Y, \xi_i) - h(Y, \nabla_X \xi_i) \\ &\quad - \nabla_Y h(X, \xi_i) + h(\nabla_Y X, \xi_i) + h(X, \nabla_Y \xi_i).\end{aligned}$$

From (3, 10), (3, 12) and (3, 13), we get

$$\bar{R}(X, Y)\xi_i = R(X, Y)\xi_i.$$

□

Corollary 2. *Let M be an invariant submanifold of the a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$\begin{aligned}(3.15) \quad R(\xi_j, X)\xi_i &= -\varphi^2 X \\ R(X, \xi_j)\xi_i &= \varphi^2 X \\ R(\xi_k, \xi_j)\xi_i &= 0\end{aligned}$$

$$(3.16) \quad R(\xi_j, X)Y = \sum_{j=1}^s \{g(X, \varphi^2 Y)\xi_j - \eta^j(Y)\varphi^2 X\}$$

for all $X, Y \in \Gamma(TM)$, $i, j, k \in \{1, 2, \dots, s\}$.

Corollary 3. *An invariant submanifold M of a generalized Kenmotsu manifold \bar{M} , ξ_i are tangent to M for all $i \in \{1, 2, \dots, s\}$, is a generalized Kenmotsu manifold.*

Example 2. *The consider a submanifold of example1 defined by*

$$M = X(u, v, w_1, w_2, w_3) = (f_1 u - f_2 v, f_1 u + f_2 v, 0, 0, w_1, w_2, w_3)$$

where f_1 and f_2 are given by

$$\begin{aligned}f_1(z_1, z_2, z_3) &= c_2 e^{-(z_1+z_2+z_3)} \cos(z_1 + z_2 + z_3) - c_1 e^{-(z_1+z_2+z_3)} \sin(z_1 + z_2 + z_3), \\ f_2(z_1, z_2, z_3) &= c_1 e^{-(z_1+z_2+z_3)} \cos(z_1 + z_2 + z_3) + c_2 e^{-(z_1+z_2+z_3)} \sin(z_1 + z_2 + z_3)\end{aligned}$$

for nonzero constant c_1, c_2 . Then local frame of TM

$$\begin{aligned}e_1 &= f_1(z_1 + z_2) \frac{\partial}{\partial x} + f_2(z_1 + z_2) \frac{\partial}{\partial y} \\ e_2 &= -f_2(z_1 + z_2) \frac{\partial}{\partial x} + f_1(z_1 + z_2) \frac{\partial}{\partial y} \\ e_3 &= \frac{\partial}{\partial z_1}, \quad e_4 = \frac{\partial}{\partial z_2}, \quad e_5 = \frac{\partial}{\partial z_3}\end{aligned}$$

We can easily that M is an invariant submanifold.

Theorem 7. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$(3.17) \quad (\nabla_X h)(Y, \xi_i) = -h(Y, \nabla_X \xi_i)$$

for all $X, Y \in \Gamma(TM)$.

Proof. Using (3, 13), then we have

$$(\nabla_X h)(Y, \xi_i) = \nabla_X h(Y, \xi_i) - h(\nabla_X Y, \xi_i) - h(Y, \nabla_X \xi_i) = -h(Y, \nabla_X \xi_i).$$

□

Corollary 4. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$(3.18) \quad (\nabla_X h)(Y, \xi_i) = -h(X, Y)$$

for all $X, Y \in \Gamma(TM)$.

Theorem 8. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then h is parallel if and only if M is totally geodesic.*

Proof. Suppose that h is parallel. Then, for each $X, Y \in \Gamma(TM)$,

$$(\nabla_X h)(Y, \xi_i) = 0.$$

Using (3, 18), we get

$$h(X, Y) = 0.$$

Vice versa, let M be totally geodesic. Then $h = 0$. For all $X, Y, Z \in \Gamma(TM)$,

$$(\nabla_X h)(Y, Z) = \nabla_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0.$$

Thus we have

$$\nabla h = 0.$$

□

Theorem 9. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$A_N \xi_i = 0$$

for all $N \in \Gamma(TM^\perp)$.

Proof. Using (3, 3) and (3, 13), we get $g(A_N \xi_i, Y) = g(h(\xi_i, Y), N) = 0$. □

Theorem 10. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$\varphi(A_N X) = A_{\varphi N} X = -A_N \varphi X$$

for all $X \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$.

Proof. By using (2, 3), (3, 3) and (3, 10) for all $X \in \Gamma(TM)$, $N \in \Gamma(TM)^\perp$ we get

$$\begin{aligned} g(\varphi(A_N X), Y) &= -g(A_N X, \varphi Y) \\ &= -g(h(X, \varphi Y), N) \\ &= -g(h(\varphi X, Y), N) \\ &= -g(A_N \varphi X, Y). \end{aligned}$$

Then, we have $\varphi(A_N X) = -A_N \varphi X$.

Moreover,

$$\begin{aligned} g(A_{\varphi N} X, Y) &= g(h(X, Y), \varphi N) \\ &= -g(\varphi(h(X, Y)), N) \\ &= -g(h(X, \varphi Y), N) \\ &= -g(A_N X, \varphi Y) \\ &= g(\varphi(A_N X), Y). \end{aligned}$$

Then, we get $A_{\varphi N} X = \varphi(A_N X)$. □

Theorem 11. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then, the second fundamental form h is η -parallel if and only if*

$$(\nabla_X h)(Y, Z) = -\sum_{i=1}^s \{\eta^i(Y)h(X, Z) + \eta^i(Z)h(X, Y)\}$$

for all $X, Y, Z \in \Gamma(TM)$.

Proof. Let M be an invariant submanifold of a generalized Kenmotsu manifold \bar{M} . The second fundamental form h of M is said to be η -parallel if $(\nabla_X h)(\varphi Y, \varphi Z) = 0$ for all vector fields X, Y and Z tangent to M .

First of all, we have

$$(\nabla_X h)(\varphi Y, \varphi Z) = \nabla_X h(\varphi Y, \varphi Z) - h(\nabla_X \varphi Y, \varphi Z) - h(\varphi Y, \nabla_X \varphi Z).$$

Then

$$\nabla_X h(\varphi Y, \varphi Z) = h(\nabla_X \varphi Y, \varphi Z) + h(\varphi Y, \nabla_X \varphi Z).$$

Using (3, 11), we get

$$\begin{aligned} -\nabla_X h(Y, Z) &= h((\nabla_X \varphi)Y + \varphi(\nabla_X Y), \varphi Z) + h(\varphi Y, (\nabla_X \varphi)Z + \varphi(\nabla_X Z)) \\ &= h((\nabla_X \varphi)Y, \varphi Z) - h(\nabla_X Y, Z) + h(\varphi Y, (\nabla_X \varphi)Z) - h(Y, \nabla_X Z). \end{aligned}$$

Thus, by using (3, 9) we have

$$-\nabla_X h(Y, Z) = -\sum_{i=1}^s \eta^i(Y)h(\varphi X, \varphi Z) - h(\nabla_X Y, Z) - \sum_{i=1}^s \eta^i(Z)h(\varphi Y, \varphi X) - h(Y, \nabla_X Z).$$

□

Theorem 12. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then $\bar{R}(X, Y)\xi_j$ is tangent to M for any $X, Y \in \Gamma(TM)$.*

Proof. For each $N_l \in \Gamma(TM)^\perp$ we have

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)\xi_j, N_l) &= \bar{g}\left(\sum_{i=1}^s \{\eta^i(Y)\varphi^2 X - \eta^i(X)\varphi^2 Y\}, N_l\right) \\ &= \bar{g}\left(\sum_{i=1}^s \eta^i(Y)\varphi^2 X, N_l\right) + \bar{g}\left(\sum_{i=1}^s \eta^i(X)\varphi^2 Y, N_l\right) \\ &= \sum_{i=1}^s \eta^i(Y) \left\{ \bar{g}\left(-X + \sum_{k=1}^s \eta^k(X)\xi_k, N_l\right) \right\} \\ &\quad + \sum_{i=1}^s \eta^i(X) \left\{ \bar{g}\left(-Y + \sum_{k=1}^s \eta^k(Y)\xi_k, N_l\right) \right\} \\ &= \sum_{i,k=1}^s \{-\bar{g}(X, N_l) + \eta^i(Y)\eta^k(X)\bar{g}(\xi_k, N_l) \\ &\quad - \bar{g}(Y, N_l) + \eta^i(Y)\eta^k(X)\bar{g}(\xi_k, N_l)\} \\ &= 0. \end{aligned}$$

□

Theorem 13. *Let M be an invariant submanifold of a $(2n + s)$ -dimensional generalized Kenmotsu manifold \bar{M} . Then we have,*

$$g(R(X, \varphi X)\varphi X, X) = \bar{g}(\bar{R}(X, \varphi X)\varphi X, X) - 2\bar{g}(h(X, X), h(X, X)).$$

Proof. First of all, we have

$$\begin{aligned} \bar{g}(\bar{R}(X, \varphi X)\varphi X, X) &= g(R(X, \varphi X)\varphi X, X) - \bar{g}(h(X, X), h(\varphi X, \varphi X)) \\ &\quad + \bar{g}(h(\varphi X, X), h(X, \varphi X)) \end{aligned}$$

From (3, 10), we get

$$\begin{aligned} g(R(X, \varphi X)\varphi X, X) &= \bar{g}(\bar{R}(X, \varphi X)\varphi X, X) + \bar{g}(h(X, X), h(\varphi X, \varphi X)) \\ &\quad - \bar{g}(\varphi h(X, X), \varphi h(X, X)). \end{aligned}$$

Using (3, 11) we have

$$g(R(X, \varphi X)\varphi X, X) = \bar{g}(\bar{R}(X, \varphi X)\varphi X, X) - 2\bar{g}(h(X, X), h(X, X)).$$

□

4. SEMIPARALLEL AND 2-SEMIPARALLEL INVARIANT SUBMANIFOLDS OF A GENERALIZED KENMOTSU MANIFOLD

Let M be a submanifold of a Riemannian manifold \bar{M} . An isometric immersion $i : M \rightarrow \bar{M}$ is *semi-parallel* if

$$\bar{R}(X, Y)h = \bar{\nabla}_X(\bar{\nabla}_Y h) - \bar{\nabla}_Y(\bar{\nabla}_X h) - \bar{\nabla}_{[X, Y]}h = 0$$

where \bar{R} is the curvature tensor of $\bar{\nabla}$ [3], where \bar{R} curvature tensor of the Van der Waerden-Bortolotti connection $\bar{\nabla}$ and h the second fundamental form.

In [1], K. Arslan and colleagues defined that M is *2-semiparallel* submanifolds if

$$R(X, Y)\nabla h = 0$$

for all vector fields X, Y tangent to M .

$\bar{\nabla}$ is the connection in $TM \oplus TM^\perp$ build with ∇ and ∇^\perp , where R (resp. R^\perp) denotes curvature tensor of the connection ∇ (resp. ∇^\perp). If R^\perp denotes the curvature tensor of ∇^\perp then,

$$(4.1) \quad (\bar{R}(X, Y)h)(Z, U) = R^\perp(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U)$$

for all vector fields X, Y, Z, U tangent to M [3]. In addition,

$$\begin{aligned} (\bar{R}(X, Y)\bar{\nabla}h)(Z, U, V) &= R^\perp(X, Y)(\bar{\nabla}h)(Z, U, V) - (\bar{\nabla}h)(R(X, Y)Z, U, V) \\ (4.2) \quad &\quad - (\bar{\nabla}h)(Z, R(X, Y)U) - (\bar{\nabla}h)(Z, U, R(X, Y)V) \end{aligned}$$

or all vector fields X, Y, Z, U, V tangent to M where $(\bar{\nabla}h)(Z, U, V) = (\bar{\nabla}_Z h)(U, V)$ [1].

Theorem 14. *Let M be an invariant submanifold of a generalized Kenmotsu manifold \bar{M} . Then M is semi-parallel if and only if M is total geodesic.*

Proof. Suppose that M is semi-parallel. Then, $\bar{R}(X, Y)h = 0$ for each $X, Y \in \Gamma(TM)$. Using (4, 1), we get

$$R^\perp(X, Y)h(Z, K) - h(R(X, Y)Z, K) - h(Z, R(X, Y)K) = 0.$$

We take $X = \xi_i$ and $K = \xi_j$ then,

$$R^\perp(\xi_i, Y)h(Z, \xi_j) - h(R(\xi_i, Y)Z, \xi_j) - h(Z, R(\xi_i, Y)\xi_j) = 0.$$

From (3, 13)

$$h(Z, R(\xi_i, Y)\xi_j) = 0.$$

Using (3, 15)

$$h(Z, Y - \sum_{t=1}^s \eta_t(X)\xi_t) = 0.$$

Then, we have

$$h(Z, Y) = 0.$$

□

Theorem 15. *Let M be an invariant submanifold of a generalized Kenmotsu manifold \bar{M} . Then M is 2-semiparallel if and only if M is total geodesic.*

Proof. Suppose that M is 2-semiparallel. Then, $\bar{R}(X, Y)\bar{\nabla}h = 0$ for each $X, Y, Z, U, V \in \Gamma(TM)$. Using (4, 2), we get

$$R^\perp(X, Y)(\bar{\nabla}h)(Z, U, V) - (\bar{\nabla}h)(R(X, Y)Z, U, V) - (\bar{\nabla}h)(Z, R(X, Y)U) - (\bar{\nabla}h)(Z, U, R(X, Y)V) = 0.$$

We take $X = \xi_i$ and $U = \xi_j$ then, we have

$$R^\perp(\xi_i, Y)(\bar{\nabla}h)(Z, \xi_j, V) - (\bar{\nabla}h)(R(\xi_i, Y)Z, \xi_j, V) - (\bar{\nabla}h)(Z, R(\xi_i, Y)\xi_j, V) - (\bar{\nabla}h)(Z, \xi_j, R(\xi_i, Y)V) = 0.$$

From (3, 11), (3, 12), (3, 13), (3, 15) and (3, 16) we get

$$\begin{aligned} (\bar{\nabla}h)(Z, \xi_j, V) &= (\bar{\nabla}_Z h)(\xi_j, V) \\ &= \nabla_Z^\perp(h(\xi_j, V)) - h(\nabla_Z \xi_j, V) - h(\xi_j, \nabla_Z V) \\ &= -h(-\varphi^2 Z, V) \\ &= -h(Z, V). \end{aligned}$$

Therefore, we have

$$\begin{aligned} (\bar{\nabla}h)(R(\xi_i, Y)Z, \xi_j, V) &= (\bar{\nabla}_{R(\xi_i, Y)Z} h)(\xi_j, V) \\ &= \nabla_{R(\xi_i, Y)Z}^\perp(h(\xi_j, V)) - h(\nabla_{R(\xi_i, Y)Z} \xi_j, V) - h(\xi_j, \nabla_{R(\xi_i, Y)Z} V) \\ &= -h(-\varphi^2 R(\xi_i, Y)Z, V) \\ &= -h(R(\xi_i, Y)Z, V) \\ &= -h\left(\sum_{l=1}^s \{g(Y, \varphi^2 Z)\xi_l - \eta^l(Z)\varphi^2 Y\}, V\right) \\ &= \sum_{l=1}^s \eta^l(Z)h(\varphi^2 Y, V) \\ &= -\sum_{l=1}^s \eta^l(Z)h(Y, V), \end{aligned}$$

$$\begin{aligned} (\bar{\nabla}h)(Z, R(\xi_i, Y)\xi_j, V) &= (\bar{\nabla}_Z h)(R(\xi_i, Y)\xi_j, V) \\ &= \nabla_Z^\perp(h(R(\xi_i, Y)\xi_j, V)) - h(\nabla_Z R(\xi_i, Y)\xi_j, V) - h(R(\xi_i, Y)\xi_j, \nabla_Z V) \\ &= \nabla_Z^\perp(h(-\varphi^2 Y, V)) - h(\nabla_Z(-\varphi^2 Y), V) - h(-\varphi^2 Y, \nabla_Z V) \end{aligned}$$

and

$$\begin{aligned}
(\bar{\nabla}h)(Z, \xi_j, R(\xi_i, Y)V) &= (\bar{\nabla}_Z h)(\xi_j, R(\xi_i, Y)V) \\
&= \nabla_Z^\perp(h(\xi_j, R(\xi_i, Y)V)) - h(\nabla_Z \xi_j, R(\xi_i, Y)V) - h(\xi_j, \nabla_Z R(\xi_i, Y)V) \\
&= -h(\nabla_Z \xi_j, R(\xi_i, Y)V) \\
&= -h(Z - \sum_{t=1}^s \eta^t(Z) \xi_t, R(\xi_i, Y)V) \\
&= -h(Z, R(\xi_i, Y)V) \\
&= -h(Z, \sum_{l=1}^s \{g(Y, \varphi^2 V) \xi_l - \eta^l(V) \varphi^2 Y\}) \\
&= \sum_{l=1}^s \eta^l(V) h(Z, \varphi^2 Y) \\
&= -\sum_{l=1}^s \eta^l(V) h(Z, Y).
\end{aligned}$$

Then, we get

$$\begin{aligned}
&R^\perp(\xi_i, Y)(-h(Z, V)) - (-\sum_{l=1}^s \eta^l(Z) h(Y, V)) - (\nabla_Z^\perp(h(-\varphi^2 Y, V))) \\
&-h(\nabla_Z(-\varphi^2 Y), V) - h(-\varphi^2 Y, \nabla_Z V) - (-\sum_{l=1}^s \eta^l(V) h(Z, Y)) \\
&= 0.
\end{aligned}$$

So we take $V = \xi_k$ then, we have

$$\begin{aligned}
h(\varphi^2 Y, \nabla_Z \xi_k) + \sum_{l=1}^s \eta^l(\xi_k) h(Z, Y) &= 0 \\
h(Y, \nabla_Z \xi_k) + h(Z, Y) &= 0 \\
h(Y, Z) &= 0.
\end{aligned}$$

□

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DEPARTMAN MATEMATİK, ÜNİVERSİTESİ OF GAZİ, 06500, ANKARA TÜRKİYE

E-mail address: `avanli@gazi.edu.tr`

E-mail address: `ramazansr@gmail.com`